*Journal of Organisational Studies and Innovation*

*Vol. 2, no.2, Summer, 2015*

# Efficient Portfolio Optimisation Using the Conditional Value at Risk Approach

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**Abstract**: Fifty years after Harry Markowitz's ground breaking work on mean variance portfolios, the problem of what asset(s) to invest in is still central to finance today, as is how to minimise risk for a given level of expected return or maximise the return for a given level of risk. Initially, standard deviation and variance were the prominent risk measures but for inherent flaws, their usage and application yielded inconsistent and unreliable results; while The Value-at-Risk (VaR) which was introduced was not subadditive. This lack of coherence led to its modification, the result being the Conditional-Value-at-Risk (CVaR) which is the expected loss exceeding the Value-at Risk (VaR), which proved to satisfy the coherence demand in addition to other advantages.

The goal of this work is to apply this CVaR method to resolve optimisation problems by minimizing risk under CVaR constraints. This research considers the proble of finding the optimal weights and the efficient frontier associated with the risk-return portfolio optimization model of a sample portfolio of 20 FTSE100 stocks using the historical prices (data) for scenario generation and makes use of the R software for the optimization process. This research has discovered that the CVaR optimization process was much more robust than the MVO model. In addition, the 1998 financial crisis had a significant impact on the result of the optimized portfolio. It also discovered that stability and validity of results increased with the volume of historical data used. Moreover, findings suggest isolating 2008 data will allow further investigation into the impact and significance of the crisis amongst others.

**Keywords:** *optimisation, CVar, MVO, portfolio, risk.*

## Introduction

The practice of optimization in modern investment management process owed so much to Harry Markowitz, whose 1952 seminal work opened a new vision in the world of asset management. The desire and drive to earn higher return on investments had been the hallmark and motivation of investors from all ages. Most investors are plagued with the common problems of i) limited resources, ii) where to invest, iii) how much to invest, iv) the ideal time to enter the market, v) when to exit the market and vi) the issue of risk which is unpredictable but real and consequential amongst other factors. This perhaps explains why hedge funds and institutional investors spend hugely on the search for techniques and tools aimed at recouping

optimal returns, minimizing risk and consequently reducing losses via rational and efficient management of capital invested.

This work delves into the age-long and critical problem of asset selection portfolio optimisation, which addresses how to allocate assets in such a way as to maximise or minimise a specific (financial) objective as the case may be. It builds on Harry Markowitz's mean-variance portfolio model, a seminar paper he published as "Portfolio Selection" in 1952. In said paper, he reiterated the relationship between risk and volatility, which can be measured by variance or standard deviation. It is a mathematical model of linear equation assuming that the total return of a portfolio can be described by using the mean return of the assets and the variance of return between these assets. For a given level of risk, one can derive the maximum return by maximizing the expected return of a portfolio or alternatively for a given specific return one can derive the minimum risk by minimizing the variance of a portfolio. It could be construed that this aided the popularity of mathematical modelling in finance and with the development of new techniques in operations research, coupled with the rapid progress in computer and software industry, lead to lots of new models based on mathematical programming being developed over the years to solve the current portfolio selection problems which involve a complex, yet realistic set of constraints.

While the model is of immense theoretical benefits, Markowitz mean-variance (MV) opimtised portfolios often fail to meet practical investment goals of marketability, usability, and performance (Michaud and Michaud 2008). This prompted many investors to seek other alternatives. Further to this, flaws in the use of variance as a measure of risk which was deemed inadequate due to its conceptual difficulties: the assumption that returns are normally distributed and that investors' utility functions made them focus on just expected return and variance. While return on investments are far from the norm as it is obvious that financial markets do not follow the logic of normal distributed returns, it has also been proven in general that an individual who maximizes expected utility may care about moments of the distribution of wealth in addition to its mean and variance (Kraus and Litzenberger, 1976). The search for the ideal measure of risk led Artzner et al. (1999) to conclude that a good risk measure must be coherent. The classical Markowitz's mean variance model was the foundation on which other later models were developed to address the highlighted inadequacies of the mean-variance approach. The mean absolute deviation (MAD) model, mean semi absolute deviation model, below target risk model and the minimax model are all different models designed as an improvement on the MVO.

As pointed out earlier, Portfolio Optimisation involves efficiently allocating the limited resources (capital) to meet desired objectives. Optimisation is a branch of applied mathematics that is widely adopted across many fields of endeavor, especially finance. Its' development and popularity is predicated on the availability of advanced algorithms for the efficient and robust solution of many of its problem classes. All optimisation problems are built on the three fulcrum of decision variables, the objective function and constraints. The application of mathematical optimisation models in finance had transformed a hitherto cumbersome and tasking process to a very sophisticated and exciting science.

Recall the fact that the future returns on assets are not known when the investment process is initiated. Hence the success of the asset selection process is hinged on the proper estimation of the risk element as the performance of any portfolio is a function of its risk because measures of risk play a crucial role in optimisation under uncertainty; especially in coping with the losses that might be incurred (Rockafellar and Uryasev 2002).

While the Markowitz's approach as a dispersion model depicted the fluctuations in earnings, it did little to show losses in investments. Due to the observation that positive and negative deviations of investment returns from their mean affects the investor's perception and financial practice, researchers and related theories showed increasing interest in quantile based risk that focuses on losses that occurs in the tail of the loss distribution. One of the most widespread quantile-based risk measures is the Value-at-Risk (VaR) which refers to the worst expected loss at a target horizon within a predetermined confidence level. Measuring this value is quick and easy and can be communicated with ease to non-technical individuals (Jorion, 1996).

Value-at-Risk (VaR) began to see an increased usage and application in finance and risk management in recent times and its' deployment as a measure of risk became popular (Dowd, 1998). However, VaR's limitation is exposed by its inability to address the issue of losses that will be suffered beyond the VaR threshold. VaR became a subject of criticism as Pritsker, (1997) submitted that VaR is inconsistent and unreliable as different methods produced different results, while Artzner et al. (1999) identifies its lack of coherence. They opined that a good measure of risk is a function of its coherence, computational ease, linearizability amongst other factors.

Rockafellar and Uryasev, (2000) came up with the Conditional Value at Risk (CVaR), an alternative measure that accounts for the losses that will be held when VaR threshold is exceeded, with a certain degree of confidence. When Rockafellar and Uryasev (2000) developed the CVaR concept and its minimisation formula, they demonstrated its effect through several case studies, including portfolio optimization and options hedging. CVaR has been found to have many computational advantages over the VaR, while maintaining consistency with the VaR by yielding the same results in cases where both were applied to normal or elliptical distributions (Embrechts et al. 2001).

Pflug (2000) established that CVaR is a coherent risk measurement due to its subadditivity, this propelled CvaR's popularity as an efficient measure of risk. CvaR has been applied in solving different optimisation problems and other relative portfolio optimisation and financial investment problems. Alexander et.al (2006) used CvaR for portfolio optimisation of financial derivatives and Andersson et al (2000) used it for credit risk measurement; while Krokhmal et al (2002) adopted CVaR constraints for a case study on optimization of a portfolio of stocks.

It is obvious that the standard method for a CVaR optimisation problem is the linear programming (LP) approach. This work adopted the original CVaR optimisation (linear programming) model of Rockafellar and Uryasev, (2000) to optimise a portfolio of 20 stocks of FTSE100 companies, an objective function and set of constraints were generated to represent the investment problem. With the R software, the optimal parameters for the decisions were then determined using the historical prices of (stocks) assets. The confidence level and number of years were varied to examine the effect of scenarios on the process since the scenario generation is a function of the number of rows and variables (the matrix column).

The rationale of this paper stems from the popularity of the MVO and its inexplicable low usage and application in the practical world of investment. According to Adrogue (2005) the MVO, though a powerful algorithm, has not yet found its place in practical asset allocation. Considering the fact that the MVO is the foundation of most optimisation models why is MVO good for theoretical and academic discourses but not practical investment purposes? And why is the CVaR preferred? The world of investment is constantly expanding, asset allocation is the fulcrum of investment practices and to select and allocate assets optimally a tested and trusted optimization model is a must. These answered question led us to embark on this research work.

The number of stock used in building the portfolio is determine by the average return of such stock over the 20 year period under consideration, stocks with negative average returns were not used. In addition to this, only companies that retained their membership of the FTSE100 status over the 20 year period were included.

The main focus of this empirical work is to opitimise a portfolio by minimising the risk objective using the CVaR optimisation model of Rockafellar and Uryasev (2000). Firstly, the paper compares a Mean Variance optimised portfolio and the CVaR optimised portfolio with a view to ascertaining the better model. Secondly, it investigates the allocation of assets (weights) in the CVaR optimisation model by comparing the risk minimisation objective to maximisation of return objective. Thereafter, the effects of varying the confidence level on the risk measure (CVaR) and the impact of increasing the period of historical data on the optimised portfolio are discussed.

## Literature Review

The concept of risk is central to the subject of Portfolio optimisation so much so that its understanding will go a long way in enabling investors, investment managers, brokers, and academics make better and informed decisions which will undoubtedly reduce incidence of losses and resultant heartaches. The confusion and controversy associated with attempts at defining risk is brought to the fore by Knight (1921. p19) when he declared:

*"Uncertainty must be taken in a sense radically distinct from the familiar notion of Risk, from which it has never been properly separated. … The essential fact is that risk means in some cases a quantity susceptible of measurement, while at other times it is something distinctly not of this character... It will appear that a measurable uncertainty, or risk proper, as we shall use the term, is so far different from an un-measurable one that it is not in effect an uncertainty at all."*

Risk is said to be at the heart of all economic activities and the need to proactively manage, minimize or eliminate it is thus very important (Harold and Kwon 2007). Since risk is associated with and classified by specific field or endeavor, the main focus of this paper is on the Financial Risk which is the variability in cash flow and market values caused by unpredictable changes in commodity prices, interest rates, exchange rates etc. and of keener concern is the Market risk a type of financial risk defined as the risk to a financial portfolio as a result of movements in market prices which in turn is determined by factors like interest rates, stock indices, commodity prices, foreign exchange rates, real estate indices, etc.

Heffernan (2005), categorised market risks as systematic risk and unsystematic risk. Systematic risk, also known as "market risk" or "un-diversifiable risk", refers to the uncertainty inherent in the entire market or entire market segment. It describes the volatility, the day-to-day fluctuations in a stock's price. Systematic risk influences a large number of assets and it cannot be reduced through diversification. Unsystematic risk which is also known as, specific, diversifiable, residual, unique or firm-specific risk describes the type of uncertainty that comes with the specific company/stock or the industry under consideration. Unlike systematic risk, it can be reduced through diversification and it influences a single company or a small group of companies.

The biggest problem faced by investors whether individual or institutional is that of risk (Sharpe 2008), occasioned mostly by market volatilities. The level of risk an investor can take becomes very important considering that there is still a strong relation between risk and expected return (Davis, n.d.). Though the said positive risk–return relationship had been a

subject of controversy amongst researchers and academics as there is no consensus on it (Rossi and Timmermann 2009). While authors like Whitelaw (1994) and Brandt and Wang (2010) found a negative relationship between these variables, nevertheless, a lot of authors provided enough evidence to the contrary (Ghysels et al 2005)

**Markowitz's Mean Variance Optimisation.**

The many benefits of investing in multiple assets has been established before Harry Markowitz published his "Portfolio selection" in 1952, attempts at controlling risk and maximising returns on investments has been a topic of great interest for investors and academics. Markowitz provided a breakthrough through his fundamental theorem of mean-variance portfolio, which holds a 'constant variance' and 'maximizes expected return' (Elton & Gruber, 1997). The ground breaking work of Markowitz (1952, 1959) dubbed the Mean Variance Optimisation (MVO) model is the landmark of modern finance theory for optimal portfolio construction, asset allocation, investment diversification, investment management and even investment performance appraisal! The reasoning was that since investors are risk averse, which is one of the assumptions of the theory, investors desires and prefers the highest expected possible return for a given standard deviation or the lowest standard deviation for a given expected return, the other assumption was that the returns from investments are normally distributed, with this he was able to show that the risk of a diversified portfolio will be lower than the individual assets in the portfolio. He also provided the relationship between the risk of a portfolio and the co-movement between individual assets in that portfolio. His work became popular and extensively used, it acted as a catalyst as others like Sharpe (1964) and Lintner (1965) further developed the theory, the Markowitz efficient frontier became a sort of launching pad for some other serious thoughts and research in the field of finance. Firstly, the Sharpe single-model model (Sharpe 1964) is a product of Markowitz's seminal work, similarly it engendered the popular optimal one-fund theorem of (Tobin1958). Tobin (1958) concluded that the optimal one-fund theorem can be accomplished where the focus is on a single optimal portfolio using a formula to account for the amount invested in that portfolio vis-à-vis the amount invested at the current risk-free rate, the resulting efficient frontier is simply a line connecting the risk-free asset to the fund of the risky assets.

The single-index model was the foundation of Elton et al. (1976, 1978) and a couple of other researchers who advanced a method of using some simple ranking criteria for optimal portfolio selection. Alexander and Resnick (1985) in their optimisation models integrated the risk associated with the estimation and the process of estimating the input parameters.

Markowitz's MV portfolio optimization model calculates the best allocation of resources while enabling the investors to incorporate their objectives and at the same time specifying their expected return or anticipated risk. The algorithm incorporating portfolio optimisers considers possible combination of assets, while allocating resources in different ratios and consequently generating different sets of investment alternatives by maximising the return or minimising the risk as earlier specified, the combination of this risk return relationship is instrumental in generating the efficient frontier.

**The Efficient Frontier**

The efficient frontier describes the relationship between the expected return from a portfolio and the volatility (standard deviation) of the said portfolio. It is the locus of risk and return

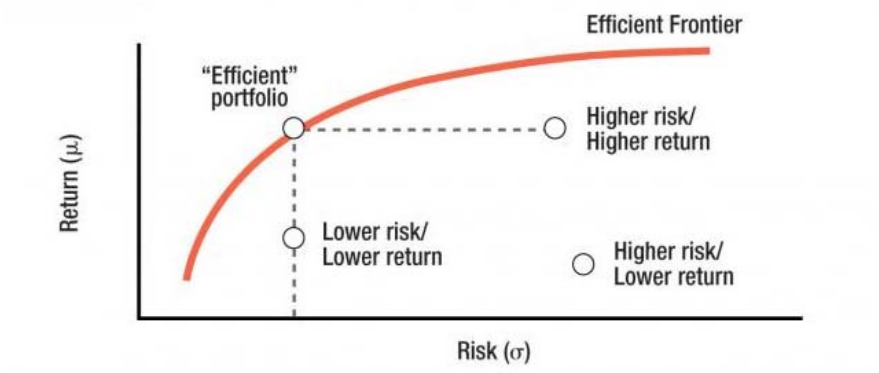
points which reflects the proportions of the equity held in the portfolio. Since expected returns and standard deviations vary given different weighted combinations of the stocks in the portfolio, when plotted, efficient frontier gives the best return that can be expected for a given level of risk or the lowest level of risk needed to achieve a given expected rate of return.

Mark Rubinstein in his lecture on Markowitz's Portfolios selection's 50-year anniversary said:

*"Probably the most important aspect of Markowitz's work was to show that it is not a security's own risk that is important to an investor, but rather the contribution the security makes to the variance of his entire portfolio- and that this was primarily a question of its covariance with all the other securities in the portfolio"* (Rubinstein 2002).

Since the goal of all Optimization problems is to find optimal solutions with respect to some constraints, in portfolio optimization, the basic assumption is that investors are risk averse i.e. if an investor is faced with multiple investments which yield similar expected returns but with different risk levels, such investor will prefer the investment with the lower risk, hence the optimisation process will be minimising risk as its objective and vice versa. A rational investor will only pick an investment with higher risk if such investment will give a commensurate higher expected return; the higher return provides a sort of incentive for him. The concept of efficient frontier describes these possible risk-return combinations that serves the investors purpose. These set of efficient portfolios lies on the efficient frontier curve in Figure 1. In the words of Drake and Fabozzi (2009), "portfolios that provide the largest possible expected return for given levels are said to be efficient".

*Figure 1: Efficient Frontier showing the risk-return combination*




As one moves up to the right, the risk increases with the return, it is left to the investor to choose the portfolio that suits his risk preference, which is his optimal portfolio. Given a choice from the set of efficient portfolios, an optimal portfolio is the one that is most preferred by the investor (Drake and Fabozzi 2009). The general acceptance of the Mean Variance model could be attributed to its application in solving the problem of asset allocation in a portfolio as this is one of the major issues in investment (Green et al, 2010).

The Markowitz's MVO model has over the years spawned innumerable theories, models and financial experiments with outstanding results especially in investment and investment performance measurement but the use of variance as risk measure is a huge drawback as it symmetrically captures both gains and losses, which might be sub-optimal. Copeland et al.

(2005) ascertained that the approach is not appropriate to describe events of low probability, which is a possibility in asset prices and returns.

Further to these were claims that the MVO model is non-intuitive and that it lacks practical applicability in the real world of investment. Michaud (1989) claimed it rendered their estimates disadvantageous. Similarly, Frankfurter et al. (1971) dealt a blow to the MVO model when they discovered that the portfolios selected using the Markowitz MV criterion is likely not as effective as an equally weighted Portfolio. Jobson and Korkie (1980) further suggest that an equally weighted portfolio will yield a superior result to an MV optimised portfolio in some instances. Different studies had tried to address the inconsistencies associated with the resulting "optimal" portfolio generated via the MVO model. Mc-Namara (1998) attributed this to the combination of returns from an unusually large number of stocks, while Michaud (1989) believed that the risk and return estimates are subjected to estimation error and that the Mean Variance optimization model significantly overweighs those securities that have large estimated returns, negative correlations and small variances and vice versa. Best and Grauer (1991) and Jorion (1992) substantiated this view as they argued that the input estimate is overly sensitive to estimation errors in the expected returns of the constituent stocks.

From the foregoing, it is apparent that the success or otherwise of an optimisation process largely depends on the accuracy of the input estimates and that of the expected return and the risk in form of standard deviation, variance or some other form of risk measure are estimated accurately, added to this mix is the correlation of these assets with each other, but there is the question of how few or large should the number of assets in an optimal portfolio be?

**How many assets makes a diversified portfolio?**

With a proper answer to this question, it could be possible pedantically estimate the number of assets to use for this project! To answer this question, different authors had over the years employed different methods resulting in myriads of contradictory conclusions and as it were, there was and there has been no definitive and uniform outcome. Evans and Archer (1968) used standard deviation as their risk measure when evaluating the reduction in portfolio risk as the portfolio size increases, they concluded that an average eight to ten stocks are sufficient to achieve most of the benefits of diversification. Fisher and Lorie (1970) found that a randomly created portfolio of 32 assets selected from the NYSE could reduce the distribution by 95%, compared to a portfolio of the entire New York Stock Exchange (NYSE) this gave birth to the much touted legend of "95% of the benefit of diversification is captured with a 30 stock portfolio." Other researchers used risk measures like variance (Elton and Gruber, 1977; Beck et al. 1996), mean absolute deviation (Fisher and Lorie, 1970) arriving at between six to fifteen stocks for a substantial level of diversification to be achieved. Depending on the wealth of investors, Ivkovic et al. (2008), arrived at about four to eleven stocks while Satman (1987) stated that a well-diversified portfolio must contain at least thirty stocks.

However, Tang (2004) opined that, these "...findings are based on the expected portfolio variances (or risk) for different portfolio sizes, which are different from the actual portfolio risk. There is no guarantee that the risk of one particular portfolio is the same as the expected risk with the same portfolio size (i.e., risk of the portfolio risk exists). Hence, there are additional sample risks in your portfolio that may not be the same as the population average". Bird and Tippet (1986) claimed the parameters used in Evans and Archer's methodology as being biased. They went on to prove that there were considerable benefits obtained by expanding the portfolio size beyond the levels suggested by Evans and Archer. While the

debate on the number of assets or stocks needed to achieve an optimal level of diversification rages on, Statman (1987), conclusively advised that "investors should only increase the number of their holdings as the marginal benefits of diversification exceeds their costs".

## Other Portfolio Optimisation Models

As earlier reiterated, Optimisation problems are to find solutions which are optimal subject to some predetermined targets, a model is then built to solve this problem, in portfolio optimisation, the investor holds a portfolio that contains different assets, bearing in mind the uncertainty surrounding his investments, he seeks to allocate more resources to those assets he believes are less risky or which will yield highest return at the expense of other deemed not to fit into this criteria.

Selecting such stocks had hitherto been a function of the investor's favourite assets, his experience and intuition, rule of thumb, or a spurious rating provided by an agent who is seen as a genius! In most cases such portfolios are suboptimal and inefficient. To solve this problem, modern optimisation models were built using sophisticated software and complex algorithms. Most of these models are still an offshoot of MVO model and the adoption of mathematical programming which includes linear programming, quadratic programming, nonlinear programming, integer programming, etc. is becoming very prevalent to Mokthar et.al (2014).

The approach adopted solving an optimisation problem and developing the optimisation model which is largely determined by the objectives and constraints of the decision variables. Optimisation falls into any of these categories:

- Linear Optimisation: One of the most common and easiest optimisation problems is the linear optimisation (LO) or the linear programming (LP) problem i.e. optimizing a linear objective function subject to linear equality and inequality constraints. The CVaR optimisation problem which is the focus of this work is a linear programming problem.

- Quadratic Optimisation: A more general optimisation problem is the quadratic optimisation (QO) or the quadratic programming (QP) problem, where the objective function is now a quadratic function of the variables, a good example is the Markowitz mean-variance portfolio optimisation problem.

In finance, variables like return on investments, risks etc. cannot be determined as at the initial time of formulating the problem and building the model as these parameters are realized in the future. When carrying out optimisation with data uncertainty, Stochastic and Robust Optimisation are the major modelling techniques for addressing such (data uncertainty).

## The Portfolio Optimisation Constraints

In other to have a more realistic result in optimisation modelling, the portfolio optimisation model must incorporate real life scenarios like the transaction costs, minimum transaction lots, cardinality constraints and thresholds on maximum or minimum investments etc. Transaction costs describes the costs of trading the assets within the portfolio in order to change the portfolio weights, this is done to maintain the optimal portfolio as this normally changes with time owing to the dynamic and random nature of the market. If care is not taken, the incentive to re-optimise frequently will lead to a build-up of such avoidable costs associated with frequent trading. Kellerer et al. (2000), showed the impact of the introduction of real features

in a portfolio optimisation model on the resulting portfolio, they discovered that the introduction of fixed transaction costs to their real data reduces the number of securities selected, and that considering transaction costs substantially changes the structure of the resulting portfolio, both in terms of securities selected and capital invested in the securities. Other constraints could be regulatory, the tax cost associated with the asset, short selling; depending on the investor's preference, as unconstrained portfolio optimisation would lead to short-selling of some assets.

In addition to the imposition of constraints to optimise portfolio selection, definition of objective, (which could be maximizing expected return or minimizing risk), the risk preference and acceptable potential losses must be properly defined and specified as these will have a telling impact on the outcome.

**Sharpe Ratio**

Measuring the risk-reward ratio in Portfolio management is important as this helps to ascertain the efficiency of a model or approach. The Sharpe ratio (Sharpe, 1966, 1994) seems to be the best suited. Roy (1952) had earlier develop a measure of risk-reward ratio, which he opined could be used to measure performance, it efficiently measures the amount of return earned per unit of risk, it can also be used in comparing investments from different asset classes. Sharpe (1966) applied and adapted Roy's concepts to Markowitz's mean-variance background resulting in what today is perhaps the best measure of performance in investment.

The use of the Sharpe Ratio has been subjected to several criticism limiting its applications and applicability as a measure of performance. In the words of Lo (2002) 'the building blocks of the Sharpe ratio—expected returns and volatilities—are unknown quantities that must be estimated statistically and are, therefore, subject to estimation error. It has since been proven that even if the normality assumption on returns themselves are dropped the Sharpe ratio will still converge toward a Normal distribution (Sharpe 2000; Prado and Peijan 2004). Bernardo and Ledoit (2000) equally contended that Sharpe ratios are misleading when the shape of the return are not normal. While modifications were made to address these shortcomings, the Sharpe Ratio in its basic raw format remains very popular amongst practitioners and academics owing to its relative simplicity and the straightforwardness of its formula in addition to its application across asset types, as it lends itself to comparism with other risk measures.

**Value-at-Risk**

Other measures of risk includes the Value-at-Risk (VaR) which today is one of the most popular concepts in Finance, VaR emerged as a concept in the 1980's and grew in prominence as a risk measure in the early 1990's. Baumol (1963) had first attempted to estimate the risk faced by financial institutions, it was J.P. Morgan (Bank) that was credited with the emergence of VaR (Linsmeier and Pearson, 1996). J.P Morgan published the VaR methodology in 1994 prompting its acceptance and adoption by financial institutions, its popularity was aided by its simplicity and its results are very stable across applications. According to Linsmeier and Pearson (1996), "Value at risk is a single, summary, statistical measure of possible portfolio losses. Specifically, value at risk is a measure of losses due to 'normal' market movements. Losses greater than the value at risk are suffered only with a specified small probability. Subject to the simplifying assumptions used in its calculation, VaR aggregates all of the risks in a portfolio into a single number suitable for use for reporting or decision making. It is simply a

way to describe the magnitude of the likely losses on the portfolio. Zhang et al (2014) stressed the "maximum possible loss", "certain quantile level" "over a given or target horizon" within a given confidence interval" in their definitions of VaR.

While each market had its own method of quantifying risk, it is difficult to adopt such models across market as each market has its own market specified criteria, Value-at-Risk (VaR) provided a solution to this by providing an integrated way of measuring different risk across different markets while combining all of the factors into a single number which serves as an indicator of the overall risk level. Dubofsky and Miller (2003) pointed out that VaR is the financial industry's standard for measuring exposure to financial price risk. The Basel Committee on Banking Supervision chose VaR as the international standard for external regulatory purposes. The growth of the software industry also helped in no small measure as sophisticated software development ensures that VaR based risk management system become more powerful thus enabling VaR systems to be used not only in market and other risks (Dowd, 2005), but also in risk supervision and reporting, it has also been deployed for extreme risk prediction, portfolio selection, risk assessment etc. (Kibzun and Kuznetsov 2006).

## Estimating the Value at Risk

VaR can be calculated by using parametric approaches i.e. Monte Carlo simulation and Variance-Covariance methods, while it is done non-parametrically with the Historical Simulation method, these are the three major approaches used in VaR estimation.

The Monte Carlo method simulates future shares/asset prices based on random normally distributed variables, a large number of randomly generated simulations are run repeatedly. Each simulation gives a possible value for the portfolio at the end of holding horizon. If a large number of simulations are used, distribution that will converge to the unknown true distribution of portfolio returns is arrived at. Then VaR could be derived from the simulated distribution of end- period portfolio values (Dowd, 2005).

The Variance-Covariance Method relies on the assumption that share returns on risk factors are normally distributed and the correlations between these risk factors are constant, similarly, the (delta) "δ" i.e. price sensitivity to changes in a risk factor of each portfolio constituent is constant. The variance and covariance of the assets are calculated and the correlation coefficient is used to isolate the volatility of each risk factor. VaR then is calculated from the Variance-Covariance matrix. Dowd (2005)) believed that the VaR is a relatively simple analytical tool for solving the risk-measurement problem but the assumptions of a normally distributed returns, constant correlations, and constant deltas are over-simplified.

The historical simulation method for calculating VaR is the simplest process, it uses historical price data to form a distribution and calculate the portfolio returns. This returns are used to calculate the worst losses for a given percentage. In practice, an hypothetical time series of returns on a portfolio is created by running the portfolio through actual historical data and computing the changes that would have occurred in each period. The changes are ranked, the chosen quantile of the profit or loss distribution is then estimated.

### *Choosing VaR Parameters*

As stated earlier, two of the parameters in estimating VaR are the holding period and the confidence level which are chosen arbitrarily, the value chosen for these quantitative factors depends largely on the application and situation in question as these determines the criteria that defines the choice of these parameters (Dowd, 2002) said "the 'best' choice of these parameters often depends on the context. They advised that where appropriate, one should work with ranges of parameter values rather than particular point values stressing that a VaR surface is much more informative than a single VaR number". When comparing risks across different markets, it is advisable to maintain consistency with the chosen figures. According to Dowd (2002), usual holding period ranges from one day to one month, but institutions can also operate on other holding periods. He said the liquidity of the markets in which the institution operates determines the length of the holding period: other things being equal, the holding period appropriate in any given market is, ideally, the length of time it takes to ensure orderly liquidation of positions in that market.

### *Challenges and Limitations of VaR*

Dowd (2002) highlighted that VaR provides a common consistent measure of risk across different positions and risk factors providing us a common risk yardstick which makes it possible for institutions to manage their risks in new ways that were not possible before but many researchers had labelled VaR 'a narrow measure of risk'. Koenker and Bassett (1978) pointed out that for a portfolio, assets' returns are assumed to be normally distributed. It has been reiterated that this assumption of normality is not always valid as the normal distribution based VaR models tend to underestimate risk McNeil et.al. (2005).

Since VaR estimation relies on the dependence and co-movement between the assets in a portfolio, Poon et al. (2003) argued that Pearson Correlation coefficient used to measure this dependence is not a very good measure of dependency. Similarly, it is believed that small scale VAR models are typically preferred because parameter estimates are much more precise thereby suggesting that large scale VaRs are unreliable. Human nature was introduced into the argument by Hoppe (1999) who described the transfer of mathematical and statistical models from physical sciences to social sciences as naïve and often invalid considering the fact that it usually ignores important features of social systems alluding to "the ways in which intelligent agents learn and react to their environment, the *non-stationarity* and dynamic interdependence of many market processes".

VaR estimates are said to be too imprecise to be of much use as empirical evidence presented by Beder (1995) and others suggests that different VaR models can give very different VaR estimates. Marshall and Siegel (1997) also showed that VaR models are exposed to considerable implementation risk and even theoretically similar models could give quite different VaR estimates because of the differences in the ways in which the models are implemented. Danielsson et al. (2003) also advanced reasons to believe that poorly thought through regulatory VaR constraints could destabilise the financial system by inducing banks to increase their risk taking. Artzner et al. (1997, 1998) agreed that though VaR offers a simple and intuitive way of evaluating market risk, but they questioned its suitability for two main reasons, they proved that VaR is not subadditive and that the measure gives only an upper limit on the losses given a specific confidence level, but it tells nothing about the potential size of the loss if this upper limit is exceeded, they proved that the VaR of a portfolio can be higher than the sum of VaRs of the individual assets in the portfolio, hence as a measure of risk VaR is described as not "coherent".

According to Artzner et al. (1999) for a measure risk to be coherent, it must satisfy the: Positive Homogeneity, Monotonicity, Risk Free Condition and the Subadditivity conditions. Artzner et al. (1999) also proved that VaR satisfied all other conditions except subadditivity. The desirability of a VaR as a subadditive risk measure lies in the diversification principle of modern portfolio theory since a subadditive measure would always result in a lower risk measure for a diversified portfolio than a non–diversified portfolio (Danielsson et.al., 2003).

### Conditional Value at Risk

The lack of subadditivity and convexity of VaR pointed out by Artzner et al (1999) led researchers to its modification, the result is the Conditional value-at-risk (CVaR) by Rockafellar and Uryasev (2000). It is derived by taking a weighted average between the value at risk and losses exceeding the value at risk. CVaR is defined by Rockafellar and Uryasev (2000) as the conditional expected loss under the condition that it exceeds VaR. It approximately (or exactly, under certain conditions) equals the average of some percentage of the worst-case loss scenarios. It is also known as "Mean Excess Loss", "Tail VaR" or "Mean Shortfall". CVaR is much more attractive than VaR since it satisfied the “coherent” flaw of VaR (Pflug, 2000). Rockafellar and Uryasev (2000, 2002) also proved that CVaR is superior to VaR in optimisation applications. Unlike VaR, CVaR quantifies the losses that might be encountered in the tail of the distribution beyond the suggested threshold, i.e. CVaR accounts for losses exceeding VaR. Further to this, Rockafellar and Uryasev, (2000) showed the effectiveness of CVaR through their work by deploying CVaR in portfolio optimisation and options hedging, they developed a minimisation formula and proved that for CVaR, once the confidence level is specified and the distribution defined, he function is convex and can lend itself to linear programming in addition to satisfying the coherence risk measure. These attributes were considered in choosing CVaR as the measure of risk for this paper.

## Methodology

This research project indeed has succeeded in opening a new vista in the quest for knowledge as it presented a veritable opportunity to adopt, modify and explore other aspects of the research topic.

Positivism paradigm is adopted because the work is scientific and experimental. In a way it is tweaking the Markowitz’s theory and the generalizations and conclusions made were based on factual observations and systematic deductions, void of personal beliefs and biases. This research satisfies all the demands and features of the deductive approach used for the research work as the researcher began with a theory; research questions were generated from the theory, leading to the main variables that dictated the type of data to be collected. An experiment and simulation process was built around the collected data which was analysed to answer the research questions.

### Data Collection

The historical prices of selected 20 equities between December 31, 1994 and December 31, 2014 were downloaded, this is about 5208 observations. The assets, with different capitalisations were chosen from different sectors of the economy in order to aid the objectivity of the experiment and the reliability of the model. The data as specified is from the FTSE100

which is an index of the 100 most highly capitalised UK companies listed on the London Stock Exchange. These stocks were selected based on the criteria described below:

i. the stocks making up the portfolio are analysed and optimmised using the (downloaded) historical data, companies with negative average returns over the period considered are excluded.

ii. Companies making up the FTSE100 groups are selected quarterly, and the FTSE is continuously monitored to include new companies and exclude some using a set of criteria, it means some Companies might be in at the beginning of the period examined and out before the end of the same period, and vice versa. Such companies are not included.

iii. Records showed that the FTSE 100 makes up for over 80 per cent of the total UK market capitalization, hence large companies are given preference over small and medium-sized companies as this is assumed to be a better reflection of the whole market.

**Method of Data Analysis**

Rockafellar and Uryasev (2000) had proved that linear programming techniques (a mathematical method) can easily be adopted for optimisation of the Conditional Value-at-Risk (CVaR) risk measures. The use of mathematically based methods, to analyse data sets the tone for quantitative research and for this work, MATLAB and R software were chosen. The *fPortfolio* package developed by Wüertz and the Rmetrics team was preferred because of its stability, versatility, popularity and the support of Diethelm Wuertz (2009) and the Rmetrics Association.

Portfolios in 'R' are defined by these three parameters: i) the data set which is usually a time series data, ii) the portfolio specification object and iii) the constraints. Listed below is the process of Optimising a CVaR portfolio:

a. a General Linear Programming Model for the CVaR Optimisation and a broad model to minimizing the CVaR risk measures was formulated.

b. the CVaR measure to input into the model formulated above was developed.

c. the formulated model(s) is then run using the dataset to create the portfolio asset allocation by weight for different levels of return.

d. the results of the portfolio allocations is then analysed, efficient frontiers are constructed, sample portfolio series, portfolio allocation graphs and the quantification of portfolio asset allocation in relation to return and risk are generated.

These steps are accomplished with a set of codes for the R system. The study was designed in such a way as to make multiple comparisons of the CVaR result across different period hence the process was done for three periods 5, 10 and 20 years estimation window, the data was saved as *STCK2020.csv, STCK2010.csv* and *STCK20005.csv* respectively for each period from where it was imported into the R environment, returns are calculated using the simple *returns()* function.

The CVaR model is represented below:

$$CVaR_\alpha = \frac{1}{1-\alpha}\int_{-\infty}^{VaR_\alpha} rp(r)dr$$

Where *p(r)* is the probability density

*r(t)* is the expected return with respect to the time horizon *t*

VaR is calculated over the same time horizon with confidence level $\alpha \in [0,1]$

## Results

### Data Analysis

The Portfolio Optimisation process was carried out with "R" a programming language and a software environment for statistical computing and graphics. This was done in a series of defined process written in the R language and the codes are executed one after the other. The process developed by Rockafellar & Uryasev (2000) which minimised CVaR was the foundation on which the CVaR Portfolio Optimisation algorithm of the *"fPortfolio package"* was built, it is a very popular package of the R software developed by Wüertz et al (2009) within the Rmetrics suite, the package comes in very handy for this Optimisation process, further credence was lend to the suitability of the approach by Pflug (2000) and Pfaff (2013).

Portfolios in 'R' are defined by these three parameters: i) the data set (*STCK2020.csv*) a time series data, ii) the portfolio specification object and iii) the constraints. To be on the safer side, the *timeSeries* package may be needed to convert data to time series using the *as.ts()* method and the date column *STCK$DATE* is properly converted to date via the *as.Date()* method or else the system will recognise the date column as numeric!

### The Portfolio Data

The historical prices series of 20 stocks from 1995 to 2014 were used and saved as *STCK2020.csv.* Returns were calculated using the simple *returns()* function. The data as specified is from the FTSE100 which is an index of the 100 most highly capitalised UK companies listed on the London Stock Exchange (Table 1).

*Table 1: The 20 companies that made up the Portfolio and their statistical summary.*

| | **Company** | **Ticker** | **Min.** | **1st Qu.** | **Median** | **Mean** | **3rd Qu.** | **Max.** |
|---|---|---|---|---|---|---|---|---|
| 1 | Associated British Foods | ABF | 154.4 | 326.9 | 563 | 729.8 | 787.6 | 3259.1 |
| 2 | Aberdeen Asset | ADN | 0 | 27.76 | 100.89 | 121.16 | 163.31 | 479.62 |
| 3 | AstraZeneca | AZN | 421.7 | 1339.3 | 1661.1 | 1793.3 | 2144.5 | 4646.8 |
| 4 | Babcock International | BAB | 1.2 | 57.93 | 90.48 | 289.33 | 431.6 | 1272.38 |
| 5 | BP | BP | 82.88 | 252.84 | 339.94 | 320.88 | 396.62 | 526.8 |
| 6 | Bunzl | BNZL | 94.41 | 215.67 | 367.95 | 496.34 | 587.79 | 1796.78 |
| 7 | Capita Group (The) | CPI | 26.26 | 201.55 | 369.26 | 431.99 | 617.78 | 1232 |
| 8 | Croda International | CRDA | 109.9 | 176.2 | 252.1 | 664.4 | 674.4 | 2659.9 |

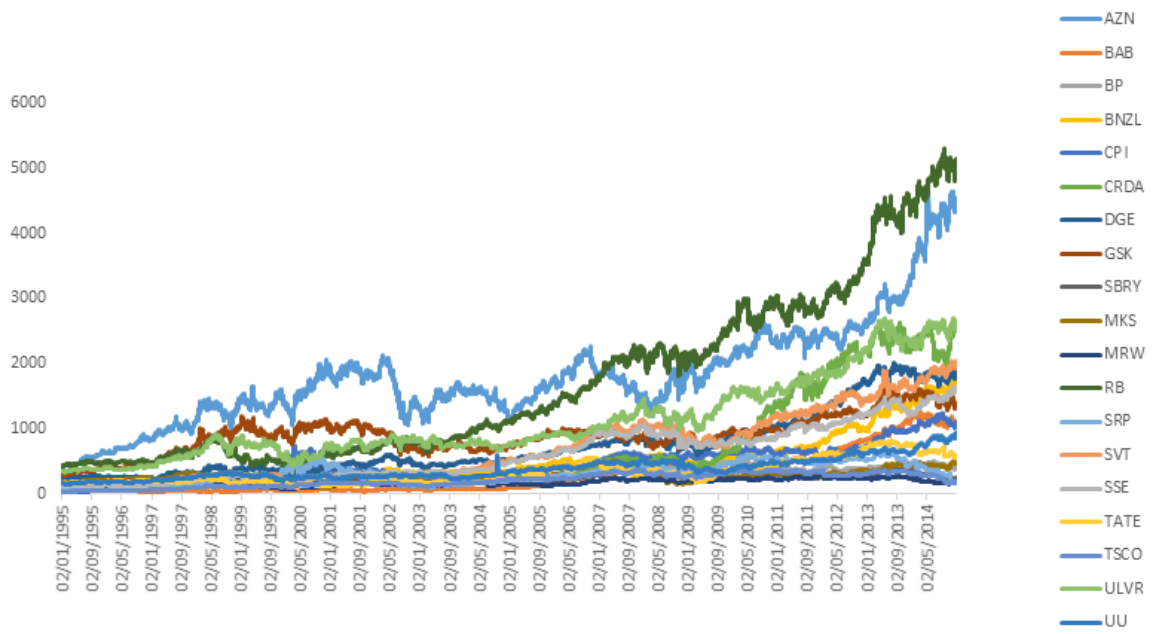
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9 | Diageo | DGE | 222.3 | 388.3 | 547.7 | 733 | 882.3 | 2020.5 |
| 10 | GlaxoSmithKline | GSK | 273.8 | 730.9 | 905.7 | 899.9 | 1025.7 | 1625.9 |
| 11 | Marks & Spencer Group | MKS | 94.58 | 196.44 | 240.54 | 268.83 | 319.55 | 516.87 |
| 12 | Morrison(Wm) | MRW | 78.96 | 129.38 | 163.42 | 171.35 | 220.22 | 279.32 |
| 13 | Reckitt Benckiser Group | RB | 297.6 | 620.8 | 1114.4 | 1678.9 | 2564.1 | 5296.4 |
| 14 | Serco Group | SRP | 38.55 | 169.64 | 303.32 | 306.03 | 432.28 | 661.72 |
| 15 | Severn Trent | SVT | 114.7 | 276 | 507.9 | 699.1 | 1033.9 | 2046.1 |
| 16 | SSE | SSE | 105.3 | 252.2 | 485.4 | 605.7 | 920.6 | 1649.3 |
| 17 | Tate & Lyle | TATE | 102.1 | 187.3 | 236.5 | 325.6 | 406.4 | 837.4 |
| 18 | Tesco | TSCO | 34.57 | 116.02 | 194.33 | 207.97 | 306.09 | 375.7 |
| 19 | Unilever | ULVR | 334.5 | 713.9 | 850.1 | 1102.7 | 1479.3 | 2708.4 |
| 20 | United Utilities Group | UU | 145.6 | 262.5 | 330.7 | 378.4 | 477.4 | 933.5 |

**Portfolio Object Specification**

Here the settings defining the CVaR Portfolio is specified:

*cvarSpec <- portfolioSpec():* this sets the Specification. *setType(cvarSpec) = "CVaR":* this Specifies the CVaR optimisation as the risk measure. *setAlpha(cvarSpec) = 0.05:* the confidence level is 0.05 by default, it can be changed to any desired level based on the target parameters of the optimisation process. The level was changed later to explore the effect of different Confidence levels on the results obtained during the optimisation process.

*Figure 2: The performance of the 20 stocks that made up the portfolio between 31/12/1994 and 31/12/2014*



The *fPortfolio* default solver interfaces are *QP quadprog*, *NLP Rdonlp2* and *LP Rglpk*. The QP quadprog is for quadratic programming, the NLP Rdonlp2 is for non-linear programming and the Rglpk_solve_LP solves linear programming (LP) and Mixed integer linear programming (MILP) programs.

The Rglpk had the read file and solve LP functions: *Rglpk_read_file* reads a linear problem from a file in either of the fixed MPS, free MPS, CPLEX, and GMPL formats, while *Rglpk_solve_LP* solves a linear and mixed linear problems. The linear programming (LP) techniques employed in the Rglpk solver aided its efficiency as the optimisation algorithms allow for solving of very large problems with millions of variables and scenarios. Also sensitivities to parameters are calculated automatically using dual variables. It should be noted however that the desired constraint will contribute to (determine) the type of solver to be selected by the user and assigned by the function *setSolver()* as done above.

***The Constraints***

While the, *fPortfolio* package provides a series of alternative constraints like short selling (unlimited short-selling), lower and upper bounds on the assets, as well as quadratic, linear equality and inequality constraints but depending on the desired constraint, the solver has to be selected by the user and assigned by the function *setSolver()* as done above under Specification setting. The CVaR subject is minimized with the three constraints below:

$$\min CVAR_{\alpha}^{\Delta}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} r_i x_i \geq \bar{r} \quad \text{=> the portfolio mean return}$$

$$\sum_{i=1}^{n} x_i = 1 \quad \text{=> budget constraint}$$

$$x_i \geq 0 \quad \text{=> no shorts constraint}$$

In reality, investors can sell ‘short’ (i.e. portfolio weights can be negative) certain stocks (securities), which mean that they can sell investments that they do not currently own (by borrowing such). For the purpose of this work, short selling is not allowed, hence the CVaR portfolio is “Long only”. Setting the constraints to *"Long Only"* will force the lower and upper bounds for the weights to zero and one, respectively i.e. negative integers are not allowed.

**The Optimisation Process**

The approach is to initially find the feasible Portfolio, by specifying the target return. Thereafter, an optimised efficient portfolio can be computed with the lowest risk for a given return. The first step is to construct an equal weights portfolio, through which a portfolio with the same returns, but with a lower covariance risk can be identified.

***Asset Returns***

The return extracted with the *returns()* method while the descriptive statistics was obtained with the *stat.desc()* method.

*Table 2: Summary Statistics of the 20 assets ordered by return.*

| | Asset | mean | SE.mean | var | std.dev | coef.var |
|---|---|---|---|---|---|---|
| **1** | **BAB** | 0.0013 | 0.0007 | 0.0024 | 0.0491 | 39.2222 |
| **2** | **AND** | 0.0007 | 0.0017 | 0.0143 | 0.1197 | 173.032 |
| **3** | **CPI** | 0.0007 | 0.0003 | 0.0005 | 0.022 | 32.1036 |
| **4** | **ABF** | 0.0006 | 0.0002 | 0.0003 | 0.0162 | 28.0324 |
| **5** | **BNZL** | 0.0006 | 0.0002 | 0.0002 | 0.0147 | 26.8832 |
| **6** | **SVT** | 0.0005 | 0.0002 | 0.0002 | 0.0153 | 28.737 |
| **7** | **CRDA** | 0.0005 | 0.0003 | 0.0003 | 0.0185 | 37.2574 |
| **8** | **SSE** | 0.0005 | 0.0002 | 0.0002 | 0.0139 | 27.4544 |
| **9** | **RB** | 0.0005 | 0.0002 | 0.0003 | 0.0164 | 34.0511 |
| **10** | **AZN** | 0.0005 | 0.0002 | 0.0003 | 0.0165 | 36.9546 |
| **11** | **ULVR** | 0.0004 | 0.0002 | 0.0002 | 0.0154 | 39.588 |
| **12** | **DGE** | 0.0004 | 0.0002 | 0.0002 | 0.0155 | 42.1371 |
| **13** | **UU** | 0.0003 | 0.0003 | 0.0004 | 0.0199 | 59.8662 |
| **14** | **TSCO** | 0.0003 | 0.0002 | 0.0003 | 0.0174 | 54.9483 |
| **15** | **BP** | 0.0003 | 0.0002 | 0.0003 | 0.0169 | 58.0319 |
| **16** | **GSK** | 0.0003 | 0.0002 | 0.0003 | 0.0159 | 55.1926 |
| **17** | **SRP** | 0.0002 | 0.0003 | 0.0005 | 0.0212 | 86.7735 |
| **18** | **TATE** | 0.0002 | 0.0003 | 0.0004 | 0.0192 | 78.9428 |
| **19** | **MKS** | 0.0002 | 0.0003 | 0.0004 | 0.0196 | 104.624 |
| **20** | **MRW** | 0.0001 | 0.0003 | 0.0004 | 0.0191 | 148.732 |

Table 2 shows the return of individual assets making up the portfolio, note that Babcock International Group with the highest return while Aberdeen Asset Management seems to be the riskiest. Morrison (Wm) Supermarkets had the lowest return. The algorithm will takes into consideration the risk and return of individual assets when optimising the portfolio.

***Equal weight Portfolio Specification***

*setWeights(cvarSpec) <- rep(1/nAssets, times = nAssets)*

*equalweightPortfolio <- feasiblePortfolio(data = retnSTCK, spec = cvarSpec, constraints = "LongOnly")*

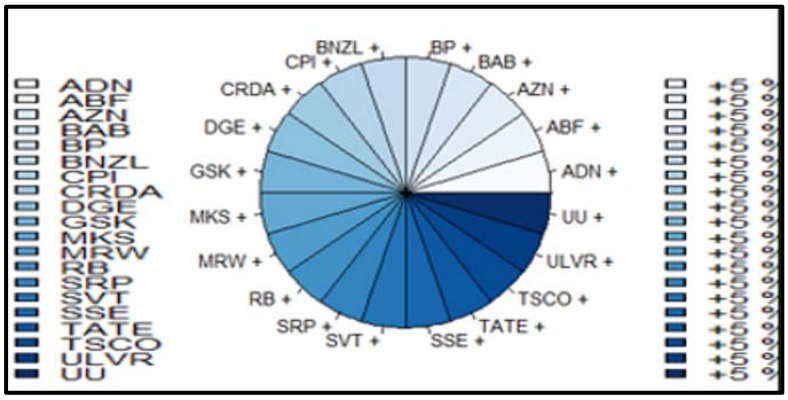
where *retnSTCK* is the generated asset returns.

*Table 3: Target Returns and Risks for equal weight portfolio.*

| mean | Cov | CVaR | VaR |
|---|---|---|---|
| | | | |

| | | | |
|---|---|---|---|
| **0.045** | **1.0689** | **2.5465** | **1.4165** |

*Figure 3: Weights plots for the CVaR Equal weight portfolio.*




Note that the weight is the same for all assets and feasible Portfolio Return is 4.5%, with the return set, finding the Minimum CVaR (Risk) Portfolio becomes easier.

***Minimum Risk Portfolio Specification***

*minriskSpec <- portfolioSpec()*

*setSolver(cvarSpec) <- "solveRglpk"*

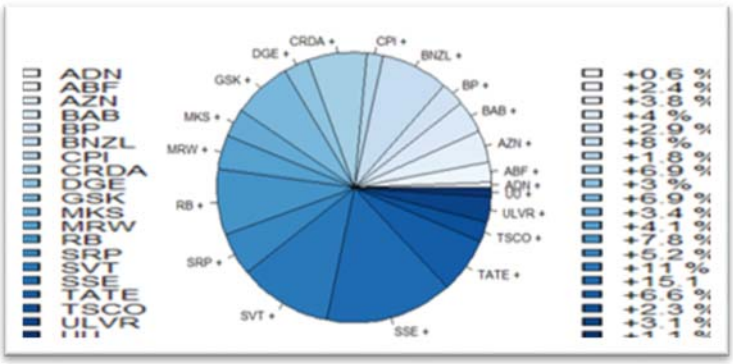
*setTargetReturn(minriskSpec)<-getTargetReturn(equalweightPortfolio)["mean"]*

Note how the return from the equal weight portfolio (*["mean"])* was called for the target return using the *setTargetReturn*() function above.

*Table 4: Target Returns and Risks for the minimum risk portfolio*

| mean | Cov | CVaR | VaR |
|---|---|---|---|
| **0.0450** | **0.8729** | **2.0027** | **1.2585** |

*Figure 4: Weights plots for the Minimum CVaR risk portfolio*



The Return is 4.5% as this is the feasible return estimated via the equal weight approach, the risk has been minimised, CVaR has fallen from **2.5465** to **2.0027** as a result of the minimisation process. Equally, worth noting is the fact that *SSE, Severn Trent and Bunzl* with the highest weights of15.1%, 11.0% and 8.0% respectively had the lowest standard deviation.

**CVaR minimised portfolio and MVO minimised portfolio.**

*Table 5: Weight and Covariance distributions of CVaR MVO minimised portfolios.*

| **Min Risk portfolio ($CVaR_{2020}$)** | | |
|---|---|---|
| **Stock** | **Weight** | **Cov** |
| **SSE** | 0.1512 | 0.1454 |
| **SVT** | 0.1097 | 0.1111 |
| **BNZL** | 0.0801 | 0.0749 |
| **RB** | 0.0776 | 0.0791 |
| **CRDA** | 0.0694 | 0.0655 |
| **GSK** | 0.0692 | 0.0679 |
| **TATE** | 0.066 | 0.0611 |
| **SRP** | 0.0519 | 0.0493 |
| **MRW** | 0.0413 | 0.0364 |
| **BAB** | 0.0396 | 0.072 |
| **AZN** | 0.0378 | 0.0365 |
| **MKS** | 0.0336 | 0.0337 |
| **ULVR** | 0.0307 | 0.0298 |
| **DGE** | 0.0301 | 0.0281 |
| **BP** | 0.0286 | 0.027 |
| **ABF** | 0.024 | 0.0224 |
| **TSCO** | 0.0234 | 0.0223 |
| **CPI** | 0.0184 | 0.0156 |
| **UU** | 0.0106 | 0.0097 |
| **ADN** | 0.0063 | 0.0122 |

| **Min Risk portfolio ($MVO_{2020}$)** | | |
|---|---|---|
| **Stock** | **Weight** | **Cov** |
| **SSE** | 0.1618 | 0.1635 |
| **BNZL** | 0.1009 | 0.1028 |
| **SVT** | 0.0928 | 0.0943 |
| **CRDA** | 0.0775 | 0.0782 |
| **TATE** | 0.0679 | 0.0653 |
| **RB** | 0.0596 | 0.06 |
| **MRW** | 0.0495 | 0.0464 |
| **CPI** | 0.0494 | 0.0516 |
| **ABF** | 0.0473 | 0.0484 |
| **AZN** | 0.0471 | 0.0471 |
| **GSK** | 0.0447 | 0.0433 |
| **DGE** | 0.0433 | 0.0426 |
| **SRP** | 0.042 | 0.0404 |
| **BP** | 0.0238 | 0.0231 |
| **ULVR** | 0.0229 | 0.0227 |
| **TSCO** | 0.0225 | 0.0219 |
| **UU** | 0.0188 | 0.0184 |
| **BAB** | 0.0164 | 0.019 |
| **MKS** | 0.0103 | 0.0098 |
| **ADN** | 0.0014 | 0.0015 |

**The output (Target Returns and Risks)**

*Table 6: Target Returns and Risks for CVaR MVO and minimised portfolios.*

| **CVaR** | | | | |
|---|---|---|---|---|
| **Period** | **mean** | **Cov** | **CVaR** | **VaR** |
| **20 yrs** | 0.045 | 0.8729 | 2.0027 | 1.2585 |
| **10 yrs** | 0.0432 | 0.8948 | 2.0827 | 1.304 |
| **5 yrs** | 0.0408 | 0.735 | 1.6311 | 1.1621 |

| **MVO** | | | | |
|---|---|---|---|---|
| **Period** | **mean** | **Cov** | **CVaR** | **VaR** |
| **20 yrs** | 0.045 | 0.8578 | 2.024 | 1.2717 |
| **10 yrs** | 0.0432 | 0.8841 | 2.1039 | 1.3066 |
| **5 yrs** | 0.0408 | 0.724 | 1.6679 | 1.1857 |

The study was designed to allow for multiple comparisons of the CVaR across 20, 10 and 5 years to be made i.e. from 1995 to 2014, 2005 to 2014 and 2009 to 2014 and as shown in the tables above the risk indices decreased over the three periods, except from the 2005 to 2014 estimation window. The covariance was also the highest for the same period for both models.

**Minimum Risk portfolio and Maximum Return portfolio compared**

The approach used for minimising risk functions under CVaR constraints was used to maximise the reward function by editing the Specification to *maxreturnPortfolio().*The return (mean) remains the same, which is the feasible return earlier estimated, while both VaR and CVaR for MVO is higher than that of CVaR optimised portfolio and Covariance between the assets is lower for MVO at 0.8578 while CVAR optimised portfolio is 0.8729.

In table 8 the return has almost double for Maximum return objective at 8.5% while Minimum Risk portfolio stood at 4.5%, this shows the investors preference in setting the input parameters to determine the output. Equally, over half of the stocks were neglected as their return were negligible to contribute meaningfully to the portfolio, the assets with highest return were given preference in contrast to the minimum risk portfolio where assets with low risk were given preference and assigned larger weight compared to those with higher risks.

*Table 7: Weight and Covariance distributions of Minimum and Maximum Return portfolios.*

| **Min Risk portfolio** | | |
|---|---|---|
| **Stock** | **Weight** | **Cov** |
| **SSE** | 0.1512 | 0.1454 |
| **SVT** | 0.1097 | 0.1111 |
| **BNZL** | 0.0801 | 0.0749 |
| **RB** | 0.0776 | 0.0791 |
| **CRDA** | 0.0694 | 0.0655 |
| **GSK** | 0.0692 | 0.0679 |
| **TATE** | 0.066 | 0.0611 |
| **SRP** | 0.0519 | 0.0493 |
| **MRW** | 0.0413 | 0.0364 |
| **BAB** | 0.0396 | 0.072 |
| **AZN** | 0.0378 | 0.0365 |
| **MKS** | 0.0336 | 0.0337 |
| **ULVR** | 0.0307 | 0.0298 |
| **DGE** | 0.0301 | 0.0281 |
| **BP** | 0.0286 | 0.027 |
| **ABF** | 0.024 | 0.0224 |
| **TSCO** | 0.0234 | 0.0223 |
| **CPI** | 0.0184 | 0.0156 |
| **UU** | 0.0106 | 0.0097 |
| **ADN** | 0.0063 | 0.0122 |

| **Max Ret portfolio** | | |
|---|---|---|
| **Stock** | **Weight** | **Cov** |
| **ADN** | 0.4293 | 0.9171 |
| **ABF** | 0.1553 | 0.0211 |
| **AZN** | 0.1223 | 0.0193 |
| **BAB** | 0.1078 | 0.0149 |
| **BP** | 0.0781 | 0.0115 |
| **BNZL** | 0.052 | 0.0064 |
| **CPI** | 0.0455 | 0.0071 |
| **CRDA** | 0.0073 | 0.0023 |
| **DGE** | 0.0024 | 0.0003 |
| **GSK** | 0 | 0 |
| **MKS** | 0 | 0 |
| **MRW** | 0 | 0 |
| **RB** | 0 | 0 |
| **SRP** | 0 | 0 |
| **SVT** | 0 | 0 |
| **SSE** | 0 | 0 |
| **TATE** | 0 | 0 |
| **TSCO** | 0 | 0 |
| **ULVR** | 0 | 0 |
| **UU** | 0 | 0 |

*Table 8: Target Returns and Risks for Minimum and Maximum Return portfolios.*

**Min Risk portfolio**

| mean | Cov | CVaR | VaR |
|---|---|---|---|
| 0.045 | 0.8729 | 2.0027 | 1.2585 |

**Max Ret portfolio**

| mean | Cov | CVaR | VaR |
|---|---|---|---|
| 0.085 | 2.2264 | 2.4983 | 1.5871 |

**Varying the estimation window and the degree of confidence**

Different CVaR constraints were imposed with different confidence levels to shape the loss distribution according to our preferences (objectives) at different estimation windows (5, 10 and 20 years).

*Table 9: The Target Returns and Risks at different estimation windows (5, 10 and 20 years) for Minimum Risk (CvaR) Portfolio*

| Period | 5 years | 10 years | 20 years |
|---|---|---|---|
| **mean** | 0.0408 | 0.0432 | 0.045 |
| **Cov** | 0.735 | 0.8948 | 1.0689 |
| **CVaR** | 1.6311 | 2.0827 | 2.5465 |
| **VaR** | 1.1621 | 1.304 | 1.4165 |

The return increase from 4.1% to 4.3% and 4.5% in 5, 10 and 20 years estimation window respectively, though other Risk return indices increased similarly, this has to do with the increase data, 20 years data definitely will produce a more reliable result than that of 5 years.

*Figure 5: The Target Returns and Risks at different estimation windows (at 5, 10 and 20 years)*

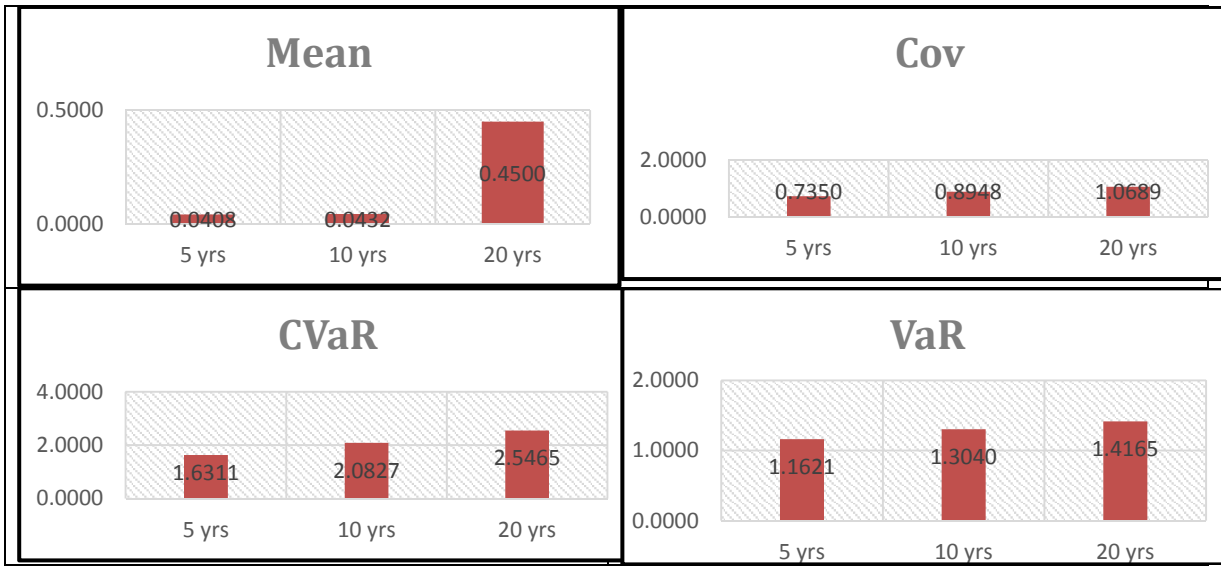



*Table 10: The Risk and Returns at different alpha levels, (0.01, 0.02, 0.05 and 0.10) for the minimum risk portfolio.*

| **Minimum Risk Efficient Portfolio (CvaR)** |
|---|

| Target Risk & Return | alpha | | | |
|---|---|---|---|---|
| | **0.01** | **0.02** | **0.05** | **0.1** |
| **mean** | 0.045 | 0.045 | 0.045 | 0.045 |
| **Cov** | 0.9159 | 0.8842 | 0.8729 | 0.8714 |
| **CVaR** | 3.2489 | 2.7188 | 2.0027 | 1.5231 |
| **VaR** | 2.4981 | 1.9763 | 1.2585 | 0.8688 |

While the return remain unchanged, risk parameters showed a consistence decrease in value with increasing confidence level.

## Conclusion

This research optimised a portfolio containing twenty assets drawn from the FTSE 100 of the UK stock market using the Conditional Value at Risk method. The performance of both CVaR and MVO optimised portfolio has been compared in order to ascertain which model is more robust. It has been established that the variance of the mean (e.g.CVaR) should be less than that of single point (e.g. VaR), and therefore the variance for CVaR should be lower than that of VaR, this is what obtains in our results hence CVaR is a more robust measure of risk.

Furthermore, the exercise revealed an unusual trend, the return was constant for both models, $CVaR_{10}$ and $MVO_{10}$ (i.e. CVaR and MVO for 10 years - 2005 to 2014) but the risk measures for that period was greater than both the 5 and 20 year periods. This difference could be attributed to some exogenous factors like the market conditions, this then is explained by the 2008 financial crisis which fell between 2005 and 2010. The impact of the crisis on the returns of the assets was significant enough to affect the (result) trend.

Based on these findings further analysis continued and the Minimum risk portfolio and the maximum return portfolio were compared. As expected the minimum risk model placed priority on the assets with low standard deviation: *SSE, Bunzl and Severn Trent* with Standard deviation of 0.0139, 0.0147, 0.0153 had the largest proportion of the weight at 15.12%, 10.97% and 8.01% respectively. Conversely, the maximum return model prioritises those assets with higher returns: *Babcock International Group, Aberdeen Asset Management, The Capita Group and the Associated British Foods* were allocated 42.3%, 15.5%, 12.2% and 10.8% respectively. This is over 80% of the total weight while others were neglected as no weights were allocated to them owing to their low return coupled with their standard deviation cum covariance with other stocks.

The confidence level of the VaR and CVaR was chosen to be between 91% and 99% for comparison, while 95% is used for other objectives (apart from comparative analysis) as these are the commonest level in literatures. The effect of varying the estimation window on the target return (mean) and the risk (CVaR) are as shown in table 9 and figure 5. While the effect of varying the confidence level on the target return (mean) and the risk (CVaR) are as shown in table 10 since the target return remains the same since it had been set (fixed), the effect is only noticeable in the Covariance, VaR and CVaR values with the increasing Confidence level.

The effect of comparing the CVaR minimised portfolio across different estimation windows revealed an increasing trend with all the indices. This is attributable to the increase in availability of more data as the period increases, the simulation and iteration is performed over a thousand more times. The observed large data samples are useful in forming the distribution and finding the Historical VaR used in the software's algorithm. The lesson here will be useful

for practitioners and financial modelling enthusiasts. The effect of data on the output of models cannot be overestimated as it increases its stability and reliability and as long as historical data will be used, the more the available data the more the positive impact on the output and reliability of the model hence a data covering 20 years will be more reliable and valid than that of 5 years.

Predicting the possible returns of assets and estimating risks associated with them and consequently that of the portfolio is a very important topic in investment as both the investors and academics are preoccupied with correct estimation of these parameters. For this research, Rockafellar and Uryasev (2000) method of Conditional value at risk to optimise a portfolio of 20 stocks taken form the FTSE100 was applied. The study allows for multiple comparisons of the CVaR and MVO methods with CVaR constrains for different time frames or estimation windows and at different confidence (alpha) levels, this not only enabled us to shape the distributions according to the investors preferences, but it also afforded us the opportunity to observe the trend of the impact of these parameters.

The CVaR methodology with the MVO approach was compared by running the optimisation algorithms on the same set of instruments, scenarios using same period and the study showed that the CVaR optimisation algorithm - which is based on linear programming techniques - is quite stable, fast and efficient using the R software. In addition, the approach can handle large numbers of assets, asset classes and scenarios effortlessly. This comparison carried out enabled us to assess the relative performance of the risk measures, using different estimation windows and confidence levels, this adds a unique dimension to the work, as it reveals a degree of state dependency in the behaviour of the risk measures. In other words these risk parameters are presumed to be affected by other factors outside the immediate factors being considered within the model i.e. the general market volatility. The ten year period comprising the year 2008 reflected this volatility effect whereas the five year period (2009-2014) i.e. post 2008 crisis did not exhibit this unusual behaviour. It was the same for the data from 1995 to 2014, in this case, the availability of large data over the 20 years (1995-2014) had smoothened out the effect of the volatility.

One other important deduction is the accuracy of large data sets, i.e. data sets if available should extend as far back into history as possible. From this experiment, it is evident that the volume of available data significantly improves the reliability of the output data, and at the same time validates the higher risk higher return cliché.

Since asset returns are non–normally distributed, as opposed to earlier assumptions, CVaR and MVO portfolio optimisation approaches reveal significant differences. In theory and practice, the traditional MVO approach results in an efficient frontier that maximizes return per unit of variance or as the case may be (depending on the investors' preference), minimizes variance for a given level of return. Whereas, the CVaR approach maximises return for a given level of CVaR or, minimizes the CVaR for a given level of determined return.

The CVaR process takes into consideration the features of the return data of the assets, in other words, assets with positive skewness, small kurtosis, and low variance are given preference. If the returns of the assets are normally distributed, both MVO and M-CVaR optimisation should lead to the same efficient frontier and, consequently, the same asset allocations.

The CVaR optimisation algorithm captures the important non-normal characteristics of the assets during the iteration process. It also make use of how those non-normal characteristics interact when combined into portfolios. The model yielded distributions that integrated

variance, skewness, and kurtosis into the CVaR estimate. In general, the results from the CVaR model showed a much more diversified and reliable weight distribution and performance.

This research optimised a portfolio using the CVaR approach, while the scope was limited to long only assets (i.e. short selling was not allowed). The research could be conducted to include assets with negative returns hence allowance could be made for short selling as this will increase the investor's opportunity to leverage on performing and non-performing assets at the same time. Although this should result in an improved performance, it will undoubtedly be good for practical investment purposes and applications, similarly, the estimation window could be increased further.

In addition, the indications of state dependency should further be investigated by isolating the 2008 data to reveal the effect and significance of the crisis and subsequent market behavior on both approaches and a comprehensive back testing process will be in order. The beauty and advantage of practical portfolio optimisation with CVaR model is captured by George Soros (1995) when he said "It's not whether you're right or wrong that's important, but how much money you make when you're right and how much you lose when you're wrong. Adopting the CVaR optimisation model will maximise this advantage.

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